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Which O-commutative Basic Algebras Are Effect Algebras

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Abstract By a *basic algebra* is meant an *MV*-like algebra $(A, \oplus, \neg, 0)$ of type $\langle 2, 1, 0\rangle$ derived in a natural way from bounded lattices having antitone involutions on their principal filters. We show that (i) atomic Archimedean basic algebras for which the operation $\oplus$ is o-commutative are effect algebras and (ii) atomic Archimedean commutative basic algebras are *MV*-algebras. This generalizes the results by Botur and Halaš on finite commutative basic algebras and complete commutative basic algebras.

Keywords Basic algebra · Commutative basic algebra · O-commutative basic algebra · Lattice effect algebra

1 Introduction

Many algebras connected with logic and foundations of physics are partially ordered sets having an antitone involution on its every principal filter. This paper is dedicated to my late colleague František Kôpka, whose thought and insight have exerted notable influence in the development of such algebras. In 1992, in the study of axiomatic system of fuzzy sets Kôpka [8] defined a new structure, a so called *D*-poset of fuzzy sets, which is closed under the formation of differences of fuzzy sets. A generalization of a *D*-poset of fuzzy sets to an abstract partially ordered set, where the basic operation is the difference were introduced in [9]. Simultaneously it was introduced an equivalent in some sense structure called effect algebra [7] as a generalization of Hilbert-space effects interpreted as the “unsharp” quantum events. Different from the “sharp” events the effects do not satisfy the non-contradiction principle, i.e., the conjunction of *a* and non *a* may be different from zero. These new logical structures for presence of propositions, properties, questions or events with fuzziness, uncertainity or unsharpness generalize orthomodular lattices (including Boolean algebras) as well as *MV*-algebras employed by C.C. Chang in the analysis of many valued logics [5].

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Definition 1 A partial algebra $(E; \oplus_E, 0, 1)$ is called an *effect algebra* if 0, 1 are two distinct elements and $\oplus_E$ is a partially defined binary operation on E which satisfy the following conditions for any $a, b, c \in E$:

(Ei) $b \oplus_E a = a \oplus_E b$ if $a \oplus b$ is defined,
(Eii) $(a \oplus_E b) \oplus_E c = a \oplus_E (b \oplus_E c)$ if one side is defined,
(Eiii) for every $a \in E$ there exists a unique $b \in E$ such that $a \oplus_E b = 1$ (we put $a' = b$),
(Eiv) if $1 \oplus_E a$ is defined then $a = 0$.

Definition 2 An MV-algebra is an algebra $A = (A, \oplus, \neg, 0)$ of type $< 2, 1, 0 >$ satisfying the identities:

(MV1)	$x \oplus (y \oplus z) = (x \oplus y) \oplus z$,	(MV2),	$x \oplus y = y \oplus x$,
(MV3)	$x \oplus 0 = x$,	(MV4)	$\neg\neg x = x$,
(MV5)	$x \oplus \neg 0 = \neg 0$,	(MV6)	$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

For a guide through the area of effect algebras and MV-algebras we can recommend [6]. Note only that MV-algebras are exactly those lattice-ordered effect algebras E in which disjoint pairs are orthogonal pairs, i.e., for all $x, y \in E$, $x \wedge y = 0$ implies $x \oplus_E y$ is defined.

As it was mentioned e.g. in [3], the mapping $x \mapsto \neg x$ is an antitone involution on $(A, \leq)$ so that $(A, \vee, \wedge, 1/4, 0, 1)$ is in fact a de Morgan algebra. Unfortunately, the reverse passage from $(A, \vee, \wedge, \neg, 0, 1)$ to $A = (A, \oplus, \neg, 0)$ is not possible due to the fact that the addition $\oplus$ cannot be expressed in terms of $\vee, \wedge, \neg$. In order to overcome this limitation, another approach was used by R. Halaš, I. Chajda and J. Kühr [4]:

Definition 3 A *lattice with section antitone involutions* is a system $L = (L, \vee, \wedge, ({}^a)_{a \in L}, 0, 1)$ where $(L, \vee, \wedge, 0, 1)$ is a bounded lattice such that every principal order filter $[a, 1]$ (called a section) possesses an antitone involution $x \mapsto x^a$. The family $({}^a)_{a \in L}$ of section antitone involutions being partial unary operations on L can be equivalently replaced by a single binary operation $\to$ defined by $x \to y := (x \vee y)^y$.

This allows one to treate lattices with section antitone involutions as total algebras $(L, \vee, \wedge, \to, 0, 1)$ or even $(L, \to, 0, 1)$ that form a variety (see e.g. [4]).

Let us recall some connections between lattices with section antitone involutions and "MV-like algebras" (for more details see again [4]).

Proposition 1

(i) *Let* $L = (A, \vee, \wedge, ({}^a)_{a \in A}, 0, 1)$ *be a lattice with section antitone involutions. Then the assigned algebra* $\mathcal{A}(L) = (L, \oplus, \neg, 0)$, *where* $x \oplus y := (x^0 \vee y)^y$ *and* $\neg x := x^0$ *satisfies the identities*

(A1)	$x \oplus 0 = x$,	(A2)	$\neg\neg x = x$,
(A3)	$x \oplus 1 = 1 \oplus x = 1$,	(A4)	$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$,
(A5)	$\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = 1$.		

(ii) *Conversely, given an algebra* $A = (A, \oplus, \neg, 0)$ *satisfying the identities* (A1)–(A5), *then for every* $a \in A$, *the mapping* $x \mapsto x^a := \neg x \oplus a$ *is an antitone involution on the section* $[a, 1]$, *and the structure* $\mathcal{L}(A) = (A, \vee, \wedge, ({}^a)_{a \in A}, 0, 1)$ *is a lattice with section antitone involutions.*
(iii) *The correspondence is one-to-one, i.e.* $\mathcal{L}(\mathcal{A}(L)) = L$ *and* $\mathcal{A}(\mathcal{L}(A)) = A$.

Definition 4 Algebras satisfying the identities (A1)–(A5) are called *basic algebras*. Given a basic algebra A and $x, y \in A$, the elements x, y are said to *commute* if $x \oplus y = y \oplus x$ holds. If every two elements of A commute then A is called a *commutative basic algebra*. A basic algebra A is called *complete* if the underlying lattice $\mathcal{L}(A)$ is complete. A is said to be a *chain basic algebra* whenever $\mathcal{L}(A)$ is a chain.

For next considerations, given a basic algebra $A = (A, \oplus, \neg, 0)$ we denote:

$$a \odot b := \neg(\neg a \oplus \neg b), \qquad a \odot_E b := \begin{cases} a \odot b & \text{if } a^0 \leq b, \\ \text{undefined} & \text{otherwise,} \end{cases}$$

$$a \to b := \neg a \oplus b, \qquad a \oplus_E b := \begin{cases} a \oplus b & \text{if } a^0 \geq b, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Similarly, we denote, for all $a, b \in A$ and all n natural, $a^1 = a$, $b^{\bar{1}} = b$, $a^{n+1} = a \odot_E a^n$ whenever $a^n \in A$ and $a \odot_E a^n$ is defined, $b^{\overline{n+1}} = b \odot b^{\bar{n}}$.

It is easy to see that $\oplus$ is commutative (associative) if and only if $\odot$ is commutative (associative). Let us list some elementary connections between $\oplus$, $\odot$, $\to$ and section antitone involutions.

$$x \oplus y = (x^0 \vee y)^y = (x^0 \odot y^0)^0 = x^0 \to y,$$

$$x \odot y = \left((x \vee y^0)^{y^0}\right)^0 = (x^0 \oplus y^0)^0 = (x \to y^0)^0,$$

$$x \to y = (x \vee y)^y = x^0 \oplus y = (x \odot y^0)^0.$$

Definition 5 Let A be a basic algebra. A is said to be *o-commutative* if $a \oplus_E b = b \oplus_E a$ holds for all $a, b \in A$ such that $b \leq a^0$. A is said to have the *block subalgebra property* if any maximal subset B of pairwise commuting elements of A called *block* is a subalgebra of A.

Let $a, b \in A$, $a^0 \leq b$. If there exists $x \in A$ with $(a \wedge b)^x = a \vee b$, we put $x = a \cdot b$ and call x the product of a and b.

Recall that the product $\cdot$ is a commutative partial operation on A. Clearly, if $a \cdot b$ exists, then $a \cdot b \leq a \wedge b$.

Let us recall some properties valid in any o-commutative basic algebra A (the proof of them will be omitted since it strictly follows the proof of this properties for commutative basic algebras contained in [2] or can be checked by an easy computation).

1. If $a, b \in A, a \leq b$, then $b^a = (a^0)^{(b^0)}$ and hence $a = (b^0 \oplus_E a) \odot_E b$.
2. Let $a, x, y \in A$, $a \geq x, y$. Then $x \geq y$ if and only if $a^x \geq a^y$. Moreover, $a^x = a^y$ if and only if $x = y$.
3. If $a, x, y \in A$ and $a \geq x, y$, then $a^{x \wedge y} = a^x \wedge a^y$ and $a^{x \vee y} = a^x \vee a^y$.
4. If $a, b \in A$ such that $b \geq a^0$, $a \cdot b$ exists and $a \cdot b = (a^{(b^0)})^0$.
5. Let $a, b \in A$. If $a \cdot b$ exists, then $a \cdot b = a \odot b$ and $b = a^{a \cdot b}$.
6. Let $x \in A$ and $a_i \in A$, $i \in I$. If the product $x \cdot \bigwedge_{i \in I} a_i$ exists, then the product $x \cdot a_i$ exists for any $i \in I$ and $x \cdot \bigwedge_{i \in I} a_i = \bigwedge_{i \in I}(x \cdot a_i)$. If the products $x \cdot a_i$ exist for all $i \in I$ and $\bigvee_{i \in I} a_i$ exists, then $x \cdot \bigvee_{i \in I} a_i = \bigvee_{i \in I}(x \cdot a_i)$.

Moreover, if A is a commutative basic algebra then

7. The induced lattice $(A, \vee, \wedge, 0, 1)$ is distributive.

8. The operations $\odot$ and $\to$ are residuated, i.e. for any $x, y, z \in A$ we have

$$x \odot y \le z \quad \text{iff} \quad x \le y \to z.$$

Therefore $\odot$ distributes over arbitrary existing joins and $\oplus$ distributes over arbitrary existing meets. Moreover, $x \odot (x \to y) = x \wedge y$ i.e. $x \odot (x^0 \oplus y) = x \wedge y$. Similarly, $x \oplus (x^0 \odot y) = x \vee y$.

Clearly, *MV*-algebras are exactly commutative and associative basic algebras. Recently, M. Botur [1] proved that there are commutative non-associative basic algebras. In [3] he and R. Halaš proved that finite commutative basic algebras are associative and hence *MV*-algebras.

The main result of our paper states that (i) atomic Archimedean basic algebras for which the operation $\oplus$ is o-commutative are effect algebras and (ii) atomic Archimedean commutative basic algebras are *MV*-algebras. Some other related results are obtained.

2 Which O-commutative Basic Algebras Are Effect Algebras

For effect algebras, *Riesz decomposition property* has been defined in [10] as follows.

For any three elements $a, b, c \in E$, $c \le a \oplus_E b$, there are $a_1, b_1 \in E$, $a_1 \le a$, $b_1 \le b$, such that $c = a_1 \oplus_E b_1$.

A lattice ordered effect algebra is an *MV*-effect algebra iff it satisfies the Riesz decomposition property [6]. There are non-lattice ordered effect algebras satisfying the Riesz decomposition property, for example the effect algebra of all polynomial functions on a real unit interval. So we will define

Definition 6 A basic algebra A has the *Riesz decomposition property* if $c, a, b \in A$ with $c \ge a \odot_E b$ imply that there exist $a_1, b_1 \in A$ with $a_1 \ge a$ and $b_1 \ge b$ such that $c = a_1 \odot_E b_1$.

Proposition 2 *Any commutative basic algebra A has the Riesz decomposition property.*

Proof Suppose that $c \ge a \odot_E b$. We put $v = c \oplus (b \vee c)^0$ and $a_1 = a \vee v$. Then $a \le a_1$, $c \le a_1$. Let us define $b_1 = a_1^0 \oplus c$. Therefore $a_1^0 \le b_1$ and $a_1 \odot_E b_1 = a_1 \odot_E (a_1^0 \oplus c) = a_1 \wedge c = c$.

We have to show that $b \le b_1$. Let us compute

$$\begin{aligned} a_1 \odot_E b_1 = c \ge (a \odot_E b) \vee (b \wedge c) &= (a \odot_E b) \vee ((c \oplus b^0) \odot_E b) \\ &= (a \odot_E b) \vee (((c \oplus b^0) \wedge (c \oplus c^0)) \odot_E b) \\ &= (a \odot_E b) \vee ((c \oplus (b \vee c)^0) \odot_E b) = (a \odot_E b) \vee (v \odot_E b) \\ &= (a \vee v) \odot_E b = a_1 \odot_E b. \end{aligned}$$

This implies $b = a_1^{a_1 \odot_E b} \le a_1^{a_1 \odot_E b_1} = b_1$. □

Lemma 1 *Let A be an o-commutative basic algebra, $a, b, x \in A$ with $a \vee x = 1 = b \vee x$ and $a^0, b^0 \le x$, $b^0 \le a$. Then $a \cdot x = a \odot x = a \wedge x$ and $a \cdot b \vee x = 1$.*

Proof Clearly, $(a \wedge x)^{a \cdot x} = a \vee x = 1$ implies $a \wedge x = 1^{a \cdot x} = a \cdot x$. Moreover by (6), $a = a \cdot 1 = a \cdot (b \vee x) = a \cdot b \vee a \cdot x$. Hence $1 = x \vee a = x \vee a \cdot b \vee a \cdot x = x \vee a \cdot b$. □

An effect algebra E is *Archimedean* if $nx = x \oplus_E x \oplus_E \cdots \oplus_E x$ (n-times) exists for every natural n only for $x = 0$. One can show that every complete effect algebra is Archimedean (see [11]).

Definition 7 A basic algebra A is said to be

1. *Archimedean* if a^n exists in A for any natural n only for $a = 1$;
2. σ*-complete* if $\bigvee_{n=1}^{\infty} a_n \in A$ for any sequence $(a_n)_{n=1}^{\infty}$ of elements of A;
3. *complete* if $\bigvee S \in A$ for any subset $S \subseteq A$.

Proposition 3 *Any σ-complete o-commutative basic algebra A is Archimedean.*

Proof Let a^n exists in A for any integer $n \geq 1$. We put $u = \bigwedge_{n=1}^{\infty} a^n \in A$. Recall that by (5), for any integer $n \geq 2$, $a^n = a \odot_E (a^0 \oplus_E a^n) = a \cdot (a^0 \oplus_E a^n)$ and $a^n = a \odot_E a^{n-1} = a \cdot a^{n-1}$. Therefore we have that $a^{n-1} = a^{a \cdot a^{n-1}} = a^{a \cdot (a^0 \oplus_E a^n)} = a^0 \oplus a^n$. This by (6) immediately implies that

$$a^0 \oplus_E u = a^0 \oplus \bigwedge_{n=1}^{\infty} a^n = \bigwedge_{n=1}^{\infty} (a^0 \oplus a^n) = \bigwedge_{n=2}^{\infty} a^{n-1} = \bigwedge_{n=1}^{\infty} a^n = u.$$

By (1) we get $a^0 = (a^0 \oplus_E u) \odot_E u^0 = u \odot_E u^0 = 0$ i.e. $a = 1$. □

Lemma 2 *Let A be an o-commutative chain basic algebra. Then*

1. *If $x \odot y > 0$ then $x \odot_E y$ exists.*
2. *A is Archimedean if and only if, for each $x, y \in A$, if $x^{\bar{n}} \geq y$ for any integer $n \geq 1$, then $x \oplus y = x$.*

Proof 1. Let $x \odot y > 0$. Then $y^0 \not\geq x$ i.e by the linearity of A we have $y^0 < x$. Therefore, $x \odot_E y$ exists.

2. Suppose that A is Archimedean and let $x^{\bar{n}} \geq y$ for any integer $n \geq 1$. If $y = 0$, then $x \oplus y = x$. Assume $0 < y$. Then $x^n = x^{\bar{n}} \geq y$ exists for any integer $n \geq 1$. Hence $x = 1$ i.e. $x \oplus y = 1 \oplus y = 1 = x$.

Conversely, let $x^n = x^{\bar{n}}$ exists for any integer $n \geq 1$. Then $x^{\bar{n}} \geq x^0$ for every $n \geq 1$. Therefore, $1 = x \oplus x^0 = x$. □

Next, we give some structural properties of commutative basic algebras.

Lemma 3 *Let A be an o-commutative basic algebra. Then, for all $a, b, c \in A$, we have: $(a \odot_E b) \odot_E c$ exists iff $a \odot_E (b \odot_E c)$ exists and $(a \odot_E b) \odot_E c = 0$ iff $a \odot_E (b \odot_E c) = 0$.*

Proof Recall that $a \odot_E b$ exists iff $b \geq a^0$. Moreover, $a \odot_E b = a \cdot b$. Similarly, $(a \odot_E b) \odot_E c$ exists iff $b \geq a^0$ and $c \geq (a \odot_E b)^0$. Evidently, $c \geq b^0$ and $c \geq (a \odot_E b)^0 = (a \vee b^0)^{(b^0)} = (a)^{(b^0)}$ i.e. $(c \odot_E b)^0 = (c \vee b^0)^{(b^0)} = c^{(b^0)} \leq a$.

Let $(a \odot_E b) \odot_E c = 0$. Then $c \leq (a \odot_E b)^0$. Therefore $c = (a \odot_E b)^0$.

We have to show that $a \odot_E (b \odot_E (a \odot_E b)^0) = 0$. It is enough to check that $a^0 = b \odot_E (a \odot_E b)^0 = b \odot (a^{(b^0)})^0$ for all $b \geq a^0$. But by (1) we have $b \odot_E (a \odot_E b)^0 = b \odot_E (a^0 \oplus_E b^0) = a^0$. The other implication can be shown by a symmetric argument. □

Definition 8 Let A be a basic algebra. A is said to be *o-associative* if, for all $a, b, c \in A$, $(a \oplus_E b) \oplus_E c = a \oplus_E (b \oplus_E c)$ whenever one side is defined.

Note that evidently A is o-associative if and only if, for all $a, b, c \in A$, $(a \odot_E b) \odot_E c = a \odot_E (b \odot_E c)$ whenever one side is defined.

We then have the following two theorems (for a comparison see [4]).

Theorem 1 *Let A be a basic algebra. Then the following conditions are equivalent*:

(i) *A is o-associative and o-commutative.*
(ii) *$(A; \oplus_E, 0, 1)$ is a lattice effect algebra.*

Proof (i) $\Rightarrow$ (ii): Evidently, (Ei) and (Eii) from the definition of an effect algebra are satisfied. Let us check (Eiii). Assume that $a \in A$. Then $a \oplus_E a^0 = (a^0)^{a^0} = 1$. Let $a \oplus_E b = 1$. Then $1 = a \oplus_E b = (a^0 \vee b)^b = (b^0 \vee a)^a$. Hence $a^0 \vee b = b$ and $b^0 \vee a = a$. This yields $a^0 \leq b \leq a^0$, therefore $a^0 = b$.

Now, let us prove (Eiv). If $1 \oplus_E a$ then $a \leq 1^0 = 0$ i.e. $a = 0$.

(ii) $\Rightarrow$ (i): This is evident since any lattice effect algebra is clearly o-associative and o-commutative. □

Theorem 2 *Let A be a commutative basic algebra. Then the following conditions are equivalent*:

(i) *A is o-associative.*
(ii) *$(A; \oplus_E, 0, 1)$ is a MV-algebra.*

Proof (i) $\Rightarrow$ (ii): By Theorem 1 is a lattice effect algebra. Since A has the Riesz decomposition property it is an MV-algebra.

(ii) $\Rightarrow$ (i): This is evident. □

Lemma 4 *Let A be an o-commutative basic algebra. Then, for all $a, b \in A$ such that $a^0 \leq b$, we have*:

$$[0, a \cdot b] \simeq [a^0, b] \simeq [b^{a^0}, 1].$$

Proof Evidently, the map $a^{(-)} : [0, a \cdot b] \to [a^0, b]$ is injective. Let $u \in [a^0, b]$. We put $v = a \cdot u$ (clearly, the product exists). Then $a^v = a^{a \cdot u} = u$. Similarly, for any $v \in [a^0, b]$, we have that $a \cdot a^v = v$.

Therefore the maps $a^{(-)} : [0, a \cdot b] \to [a^0, b]$ and $a \cdot (-) : [a^0, b] \to [0, a \cdot b]$ are mutual inverse.

Applying the same arguments, we have that

1. the maps $b^{(-)} : [a^0, b] \to [b^{(a^0)}, 1]$ and $b \cdot (-) : [b^{(a^0)}, 1] \to [a^0, b]$ are mutual inverse,
2. the maps $(a \cdot b)^{(-)} : [0, a \cdot b] \to [b^{(a^0)}, 1]$ and $(a \cdot b) \cdot (-) : [b^{(a^0)}, 1] \to [0, a \cdot b]$ are mutual inverse.

□

Corollary 1 *Let A be an o-commutative basic algebra. Then, for all $a, b \in A$ such that $a^0 \leq b$, the maps*

$$h_a^b = (a \cdot b)^{a \cdot (b \cdot (-))} : [b^{(a^0)}, 1] \to [b^{(a^0)}, 1]$$

and

$$g_a^b = (b)^{(a^{(a\cdot b)\cdot(-)})} : [b^{(a^0)}, 1] \to [b^{(a^0)}, 1]$$

are mutual inverse order-preserving bijections.

Theorem 3 *Let A be an o-commutative basic algebra. Then A is a lattice effect algebra iff for all $a, b \in A$ with $a^0 \le b$, the maps h_a^b and $\mathrm{id}_{[b^{(a^0)},1]}$ coincide. Moreover if A is commutative basic algebra, A is an MV-algebra iff A is a lattice effect algebra.*

Proof Clearly, $h_a^b = \mathrm{id}_{[b^{(a^0)},1]}$ for all $a, b \in A$ such that $a^0 \le b$ is evidently equivalent with $(x \odot_E y) \odot_E z = x \odot_E (y \odot_E z)$ whenever one side is defined. The statement then follows from Theorems 1 and 2. □

Recall that an element a of a poset P is an *atom* if $0 \le b < a$ implies $b = 0$ and P is called *atomic* if for every nonzero element $x \in P$ there is an atom a of P with $a \le x$. *Coatoms* (*maximal elements*) are defined dually. We shall denote by $\mathrm{Max}(P)$ the set of all maximal elements of P.

Proposition 4 *Let A be an atomic Archimedean o-commutative basic algebra. Then, for any $x \in A$,*

$$x = \bigwedge \{a^{n_a(x)} : a \in \mathrm{Max}(A), x \le a\};$$

here $n_a(x) = \sup\{n \in \mathbf{N} : a^n \ge x\}$.

Proof We may assume that $x \ne 1$ (this implies $n_a(x) \in \mathbf{N}$). Clearly, x is a lower bound of the set $\{a^{n_a(x)} : a \in \mathrm{Max}(A), x \le a\}$. Let y be another lower bound of $\{a^{n_a(x)} : a \in \mathrm{Max}(A), x \le a\}$. We will show that $x \vee y = x$. If $x \oplus_E (x \vee y)^0 = 1$ then $x^0 = (x \vee y)^0$. Hence $x \vee y = x$. If $x \oplus_E (x \vee y^0) < 1$ then there exists a dual atom $m \in \mathrm{Max}(A)$ such that $x \oplus_E (x \vee y)^0 \le m$. We have $x \vee y \le m^{n_m(x)}$ and by (1)

$$x = (x \vee y) \odot_E (x \oplus_E (x \vee y)^0) \le m^{n_m(x)} \odot_E m = m^{n_m(x)+1},$$

a contradiction. □

Lemma 5 *Let A be an atomic Archimedean o-commutative basic algebra. Then, for any $a \in \mathrm{Max}(A)$, $a^{m+n} = a^m \odot_E a^n$ if a^{m+n} or $a^m \cdot a^n$ exists.*

Proof We may assume that $m, n \ge 2$ (the case $m = 1$ or $n = 1$ is evident). Clearly, $a^m \cdot a^n \le a^{\max(m,n)}$. Let $a^m \cdot a^n \le b^p \ne 1$ for some $b \in \mathrm{Max}(A)$, $b \ne a$ and $p \ge 1$. Then, by a successive application of Lemma 1, $a^m \vee b^p = 1$ and $a^n \vee b^p = 1$. Again by Lemma 1 $b^p = a^m \cdot a^n \vee b^p = 1$, a contradiction. Therefore, $a^m \cdot a^n = a^k$ for some integer $k \ge \max(m, n)$ and this yields $a^n = (a^m)^{(a^k)}$. Clearly, $a^{n_a(0)} \vee b^p = 1$ for all integers $p \ge 1$. Hence any element x from the principal filter $[a^{n_a(0)}, 1]$ is of the form a^l for some integer $l \ge 1$. This immediately implies that $C = [a^{n_a(0)}, 1]$ is a finite chain and it is well known that there is a unique way of defining antitone involution on its sections and that the corresponding basic algebra is both commutative and associative, hence an MV-algebra.

Let us denote $\cdot_C$ the induced partial product operation on C. For all $x, y \in C$ we then have: $x \cdot y$ is defined iff $x \ge y^0$ and $y = x^u$ for some $u \ge a^{n_a(0)}$ iff $y = x^u$ for some $u \ge a^{n_a(0)}$ iff $x \ge y^{(a^{n_a(0)})}$ and $y = x^u$ for some $u \ge a^{n_a(0)}$ iff $x \cdot_C y$ is defined.

Assume now that $x \cdot y \in C$ is defined. Then by (5) $x^{x \cdot y} = y = x^{x \cdot_C y}$. By (2) we obtain that $x \cdot y = x \cdot_C y$. Hence a^m computed in E coincides with the corresponding product computed in C. Therefore $a^m \cdot a^n = a^m \odot_{[a^{n_a(0)},1]} a^n = a^{m+n}$. □

Let us state our main theorem.

Theorem 4 *Every atomic Archimedean o-commutative basic algebra A is o-associative and therefore a lattice effect algebra.*

Proof Let $a, b, c \in \mathrm{Max}(A)$, $m, n, p \geq 0$ integers such that $(a^m \odot_E b^n) \odot_E c^p$ exists. In what follows we will repeatedly apply Lemma 1. If $a = b = c$ then clearly $a^m \cdot (b^n \cdot c^p) = a^{m+n+p} = (a^m \cdot b^n) \cdot c^p$. Let $a = b$, $a \neq c$. Then we have that $(a^m \cdot a^n) \vee c^p = 1$ and this implies $(a^m \cdot a^n) \cdot c^p = (a^m \cdot a^n) \wedge c^p$. Similarly $a^n \vee c^p = 1$ implies $a^m \cdot (a^n \cdot c^p) = a^m \cdot (a^n \wedge c^p) = (a^m \cdot a^n) \wedge (a^m \cdot c^p) = (a^m \cdot a^n) \wedge (a^m \wedge c^p) = (a^m \cdot a^n) \wedge c^p$. Let $a \neq b \neq c \neq a$. Then $a^m \cdot (b^n \cdot c^p) = a^m \cdot (b^n \wedge c^p) = a^m \wedge (b^n \wedge c^p)$ since $a^m \vee (b^n \wedge c^p) = 1$ and $a^m \geq (b^n \wedge c^p)^0$. Similarly $(a^m \cdot b^n) \cdot c^p = (a^m \wedge b^n) \wedge c^p$. The remaining cases can be shown by a symmetric argument. Therefore $a^m \odot_E (b^n \odot_E c^p) = (a^m \odot_E b^n) \odot_E c^p$.

Let $x, y, z \in A$ such that $(x \odot_E y) \odot_E z$ exists. Then

$$\begin{aligned}
(x \odot_E y) \odot_E z &= (x \odot_E y) \odot_E \bigwedge \{c^{n_c(z)} : c \in \mathrm{Max}(A)\} \\
&= \bigwedge \{(x \odot_E y) \odot_E c^{n_c(z)} : c \in \mathrm{Max}(A)\} \\
&= \bigwedge \left\{ \left(x \odot_E \bigwedge \{b^{n_b(y)} : b \in \mathrm{Max}(A)\} \right) \odot_E c^{n_c(z)} : c \in \mathrm{Max}(A) \right\} \\
&= \bigwedge \{(x \odot_E b^{n_b(y)}) \odot_E c^{n_c(z)} : b, c \in \mathrm{Max}(A)\} \\
&= \bigwedge \left\{ \left(\bigwedge \{a^{n_a(x)} : a \in \mathrm{Max}(A)\} \odot_E b^{n_b(y)} \right) \odot_E c^{n_c(z)} : b, c \in \mathrm{Max}(A) \right\} \\
&= \bigwedge \{(a^{n_a(x)} \odot_E b^{n_b(y)}) \odot_E c^{n_c(z)} : a, b, c \in \mathrm{Max}(A)\} \\
&= \bigwedge \{a^{n_a(x)} \odot_E (b^{n_b(y)} \odot_E c^{n_c(z)}) : a, b, c \in \mathrm{Max}(A)\} \\
&= x \odot_E (y \odot_E z) \quad \text{by the same arguments.}
\end{aligned}$$

From Theorem 1 we get that A is a lattice effect algebra. □

Corollary 2 *Any atomic complete (σ-complete) o-commutative basic algebra is a lattice effect algebra.*

As a common corollary of Theorems 2 and 4 we have the following theorem.

Theorem 5 *Every atomic Archimedean commutative basic algebra A is an MV-algebra.*

Corollary 3 *Any atomic complete (σ-complete) commutative basic algebra is an MV-algebra.*

Concluding Remarks

In [4] the authors assumed the block subalgebra property to characterize basic algebras that are lattice effect algebras. In our Theorems 1 and 4 we use only the assumption that basic algebra is o-commutative. Does o-commutativity imply the block subalgebra property?

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